Complex Numbers

Complex numbers are an extension of the real numbers we all know and love, mostly for mathematical convenience. They are very useful, however, to describe how things behave (they are of importance in physics to describe the behaviour of non-measurable things) on a non-real scale. I know that sounds all a little hokey, but let's see if we can get some use from it. The mathematical convention for a complex number is the sum of two numbers, one **real**, and one **imaginary**. An imaginary number is a real number that has been mutltiplied by i (sometimes denoted as j), where

$$i = \sqrt{-1} \quad \text{so that} \quad i^2 = -1 \tag{1}$$

Mathematically, we define the complex numbers as the sum of a the real and imaginary parts

$$\mathbb{C} \equiv \{a + ib \; \forall \;\; a, b \in \mathbb{R}, \; i = \sqrt{-1}\}$$

With a complex number $z = a + ib$, the **real part** is $\Re(z) = a$, and the **imaginary part** is $\Im(z) = b$.

Every complex number has a **complex conjugate**, which is identical, with only the sign in front of all the imaginary terms changed. If a complex number looks like this

$$z = a + ib - \frac{ix}{12}$$

The it's complex conjugate (denoted z^*) would be

$$z^* = a - ib + \frac{ix}{12}$$

Notice that when you multiply z with z^*, you will get a strictly real number. Let's do that now

$$zz^* = |z|^2 = (a + ib)(a - ib) = a^2 + iab - iab - i^2b^2 = a^2 + b^2$$
$$|z| = \sqrt{a^2 + b^2} \tag{2}$$

As from (1), $i^2 = -1$. This is useful for obtaining real (as in, real life) solutions from complex numbers.

Problems

1. For a complex number $z = a + ib$, find the following

   (a) z^2
   Note: this is *not* the magnitude equation above - it's not going to look pretty.

   (b) $\frac{z}{z^*}$

2. The following quadratic equation has imaginary roots. Find the complex conjugate pair of roots for the equation (Hint: use the quadratic fomula).

$$f(x) = x^2 + x + 1$$

3. **Euler's Relation** relates complex exponents with trigonometric functions in the following way

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

   (a) Show that this equation follows the complex conjugate rules (that is, multiply both sides by its complex conjugate to show that they are equal. Remember that $\sin^2(x) + \cos^2(x) = 1$).

   (b) Using Euler's relation, and what you know about trig functions, show that

$$e^{i\pi} = -1$$

4. In Quantum Mechanics, solutions to the Schrödinger equation yield equations called wave functions, which are denoted as $\psi(x)$. The probability distribution (something you can actually work with) is given as $|\psi(x)|^2$. For the following wave equation, find $|\psi(x)|^2$

$$\psi(x) = \frac{m\omega}{\hbar} x e^{-ix^2}$$