Completing the Square

One very useful technique of algebra, especially with a second order (highest power is two) polynomial, is the method of completing the square. It makes more advanced math techniques like integration easier, and it's a good way of rearranging quadratics to make things look more intuitive.
Let's start with a general quadratic equation:

$$ax^2 + bx + c = f(x) \tag{1}$$

Note here that I can generally take any second order quadratic equation and put it into this form. It may be that the coefficients are not as simple as a number, but so long as they're constant in your equation, you're home free. If the equation is not equal to zero and you wish it to be, just bring terms over and recombine them into your new c, b, or a (depending on what's on the right hand side of your eqation). Let's focus on completing the square when $f(x) = 0$, which is the most common situation for completing the square. Now, in order to have a quadratic equation, a must not be zero (or we would not have the x^2 term), and so we can divide through by a[1]

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{2}$$

Now, we have some options from here on how to proceed, notation wise. I'm going to redefine $\frac{b}{a}$ as β, and $\frac{c}{a}$ as γ. You can *always* do this with a quadratic equation, and it can make visualizing the next step a whole lot simpler. So, now I'm left with

$$x^2 + \beta x + \gamma = 0 \tag{3}$$

Now here is where the trick comes from. You need to *add*, and *subtract* a strategic number from the equation (thus adding 0 in total and not affecting the overall equation) to make the number a perfect square. That number is

$$\left(\frac{\beta}{2}\right)^2$$

Thus your equation becomes

$$x^2 + \beta x + \left(\frac{\beta}{2}\right)^2 - \left(\frac{\beta}{2}\right)^2 + \gamma = 0 \tag{4}$$

Notice, now, that the first three terms are the expanded terms of a perfect square (you have, essentially *completed the criteria for a perfect square*). You can check for yourself, but I can write the equation above (essentially factoring the first three terms) as

$$\left(x + \frac{\beta}{2}\right)^2 - \left(\frac{\beta}{2}\right)^2 + \gamma = 0 \tag{5}$$

You have now completed the square. From here, you can go back to your bigger problem (if you have one), or you can solve for x. You can now also get a better feel for what the parabola you are dealing with looks like.

Example
Complete the square for $f(x) = x^2 + 8x + 19$.
Solution
$f(x) = x^2+8x+19 = x^2+8x+\left(\frac{8}{2}\right)^2-\left(\frac{8}{2}\right)^2+19 = x^2+8x+16-16+19 = (x^2+8x+16)+3 = (x+4)^2+3$
And so we have completed the square: $f(x) = (x + 4)^2 + 3$

[1] When you're trying to find roots, you can set $f(x) = 0$. If you are going to be keeping it as $f(x)$, then you cannot just divide through by a, and you'll be left with a slightly more complicated polynomial. The technique is the same.

Problems

1. Check for yourself that (5) really does follow from (4).

2. Confirm that you can use a similar technique for the equation $x^2 - \beta x + \gamma$, with the only difference being the sign inside the squared bracket.

3. Complete the square for the following equations

   (a) $2x^2 + 4x + 1 = 0$

   (b) $f(x) = x^2 + 2\sqrt{2}x + 6$

   (c) $f(x) = x^2 + \frac{2}{7}x + \frac{29}{49}$

   (d) $f(x) = x^2 + 6x - 1$

   (e) $f(x) = x^2 - 2x + 4$

4. The **Quadratic Formula** is a useful and powerful tool to solve for the roots of a second order polynomial, despite what constants you may have. It states that for equation (1), The roots of the polynomial are found at

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   Using the method of completing the square, derive the quadratic formula from the equation given in (1) (Hint: leave it in terms of a, b, and c, and work from equation (2). After you complete the square, you will need to use some clever algebra to isolate for x, and get it in the form above).

5. Complete the square on the following equation to determine the parabola's vertical and horizontal shift, and whether it opens upwards or downwards.

$$f(x) = x^2 - 4x + 5$$

6. Complete the square for the following equation both for x and for y, keeping them separate. The result should be a circle equation. Find the centre point, and the radius.

$$4x^2 + 4y^2 - 24x + 32y - 4 = 0$$

7. (**Integral Calculus Required**) Completing the square can make integrals a lot simpler. Try using the complete the square technique to find

$$f(x) = \int \frac{1}{x^2 + 4x + 4}\, dx$$